Aequat. Math. 89 (2015), 591–603

0001-9054/15/030591-13
published online December 6, 2013
DOI 10.1007/s00010-013-0244-4

Aequationes Mathematicae

Some generalization of Cauchy's and Wilson's functional equations on abelian groups

Radosław Łukasik

Abstract. We find the solutions $f, g, h\colon G \to X, \alpha\colon G \to \mathbb{K}$ of the functional equation

$$\sum_{\lambda \in K} f(x + \lambda y) = |K|g(x) + \alpha(x)h(y), \quad x, y \in G,$$

where $(G, +)$ is an abelian group, K is a finite, abelian subgroup of the automorphism group of G, X is a linear space over the field $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$.

Mathematics Subject Classification (2010). 39B52.

Keywords. Wilson's functional equation, Cauchy's functional equation.

1. Introduction

The following generalization

$$f(x + y) + f(x + \sigma y) = 2f(x) + 2f(y), \quad x, y \in G,$$

of the quadratic functional equation, where σ is an automorphism of an abelian group G such that $\sigma \circ \sigma = id_G$ and $f\colon G \to \mathbb{C}$, was investigated by Stetkær [9].

In his other work [10] he solved the functional equation

$$\frac{1}{N} \sum_{n=0}^{N-1} f(z + \omega^n \zeta) = f(z) + g(z)h(\zeta), \quad z, \zeta \in \mathbb{C},$$

where $N \in \{2, 3, \ldots\}, \omega$ is a primitive N^{th} root of unity, $f, g, h\colon \mathbb{C} \to \mathbb{C}$ are continuous.

Łukasik [5,6] derived explicit formulas for the solutions of the functional equation

$$\sum_{\lambda \in K} f(x + \lambda y) = |K|\alpha(y)g(x) + |K|h(y), \quad x, y \in G,$$

where $(G,+)$ is an abelian group, K is a finite abelian subgroup of the automorphism group of G, X is a linear space over the field $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}, f, g, h\colon G \to X$, $\alpha\colon G \to \mathbb{K}$.

The functional equation

$$\sum_{\lambda \in K} f(x + \lambda y) = |K| g(x) h(y), \quad x, y \in G,$$

where $(G,+)$ is an abelian group, K is a finite subgroup of the automorphism group on $G, f, g, h\colon G \to \mathbb{C}$, was studied by Förg-Rob and Schwaiger [3], Gajda [4], Stetkær [7,8], Badora [2].

Aczél et al. [1] studied the complex-valued solutions of the equation

$$f(x + y) + f(x - y) - 2f(x) = g(x)h(y), \quad x, y \in G,$$

where $(G,+)$ is a group and $f, g, h\colon G \to \mathbb{C}$.

The purpose of this paper is to find the solutions of the functional equation

$$\sum_{\lambda \in K} f(x + \lambda y) = |K| g(x) + \alpha(x) h(y), \quad x, y \in G,$$

where $f, g, h\colon G \to X, \alpha\colon G \to \mathbb{K}, (G,+)$ is an abelian group, K is a finite, abelian subgroup of the automorphism group of G, X is a linear space over the field $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$.

We find these solutions under the assumption $\sum_{\lambda \in K} f \circ \lambda \neq const, \sum_{\lambda \in K} \alpha \circ \lambda \neq const, \alpha(0) \neq 0$ and they are some combinations of multiplicative and multi-additive functions.

Our results generalize all the results mentioned above (except the papers by Łukasik).

2. Main result

Throughout the present paper, we assume that X is a linear space over the field $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$,$(G,+)$ is an abelian group, K is a finite, abelian subgroup of the automorphism group of G.

In this work we use some theorems. The first gives us the form of the solutions of a generalization of Jensen's functional equation.

Theorem 1. [5, Theorem 5] *Let $(S,+)$ be an abelian semigroup, K be a finite subgroup of the automorphism group of $S, (H,+)$ be an abelian group uniquely divisible by $|K|!$. Then the function $f\colon S \to H$ satisfies the equation*

$$\sum_{\lambda \in K} f(x + \lambda y) = |K| f(x), \quad x, y \in S \tag{1}$$

if and only if there exist k-additive, symmetric mappings $A_k\colon S^k \to H$, $k \in \{1,\dots,|K|-1\}$ *and* $A_0 \in H$ *such that*

$$f(x) = A_0 + A_1(x) + \cdots + A_{|K|-1}(x,\dots,x), \quad x \in S,$$
$$\sum_{\lambda\in K} A_k(x,\dots,x,\underbrace{\lambda y,\dots,\lambda y}_{i}) = 0, \quad x,y \in S,\ 1 \le i \le k \le |K|-1.$$

The second theorem shows all solutions of a generalization of Wilson's functional equation.

Theorem 2. [6, Theorem 4,5] *Let* $f\colon G \to X, f \neq 0, \varphi\colon G \to \mathbb{K}$. *They satisfy the equation*

$$\sum_{\lambda\in K} f(x+\lambda y) = |K|\varphi(y)f(x), \quad x,y \in G, \tag{2}$$

if and only if there exists a homomorphism $m\colon G \to \mathbb{C}^*$, *such that*

$$\varphi(x) = \frac{1}{|K|}\sum_{\lambda\in K} m(\lambda x), \quad x \in G,$$

and

(i) *if* X *is complex, then there exist* $A_0^\lambda \in X$, *k-additive, symmetric mappings* $A_k^\lambda\colon G^k \to X, k \in \{1,\dots,|K_0|-1\}, \lambda \in K_1$ *such that*

$$f(x) = \sum_{\lambda\in K_1} m(\lambda x)\left[A_0^\lambda + \sum_{i=1}^{|K_0|-1} A_i^\lambda(x,\dots,x)\right], \quad x \in G,$$
$$\sum_{\mu\in K_0} A_k^\lambda(x,\dots,x,\underbrace{\mu y,\dots,\mu y}_{i}) = 0, \quad x,y \in G, \lambda\in K_1,\ 1\le i\le k \le |K_0|-1,$$

(ii) *if* X *is real, then there exist* $A_0^\lambda \in X, B_0^\lambda \in X$, *k-additive, symmetric mappings* $A_k^\lambda, B_k^\lambda\colon G^k \to X, k \in \{1,\dots,|K_0|-1\}, \lambda \in K_1$ *such that*

$$f(x) = \sum_{\lambda\in K_1}\Bigg(Re\,(m(\lambda x))\left[A_0^\lambda + \sum_{i=1}^{|K_0|-1} A_i^\lambda(x,\dots,x)\right]$$
$$-Im\,(m(\lambda x))\left[B_0^\lambda + \sum_{i=1}^{|K_0|-1} B_i^\lambda(x,\dots,x)\right]\Bigg), \quad x \in G,$$
$$\sum_{\mu\in K_0} A_k^\lambda(x,\dots,x,\underbrace{\mu y,\dots,\mu y}_{i}) = 0, \quad x,y \in G, \lambda \in K_1, 1\le i\le k\le |K_0|-1,$$
$$\sum_{\mu\in K_0} B_k^\lambda(x,\dots,x,\underbrace{\mu y,\dots,\mu y}_{i}) = 0, \quad x,y \in G, \lambda\in K_1,\ 1\le i\le k\le |K_0|-1,$$

where $K_0 := \{\lambda \in K :\ m \circ \lambda = m\}, K_1$ *is the set of representatives of cosets of the quotient group* K/K_0.

First we start with a corollary of Theorem 2.

Corollary 1. *A nonzero function $\alpha\colon G \to \mathbb{K}$ satisfies the equation*

$$\sum_{\lambda\in K} \alpha(x+\lambda y) = \alpha(x)\sum_{\lambda\in K}\alpha(\lambda y), \quad x,y\in G, \tag{3}$$

if and only if there exists a homomorphism $m\colon G\to\mathbb{C}^*$ and

(i) *if $\mathbb{K} = \mathbb{C}$, then there exist $a_0^\lambda \in \mathbb{C}$, k-additive, symmetric mappings $a_k^\lambda\colon G^k \to \mathbb{C}, k\in\{1,\dots,|K_0|-1\}$,$\lambda\in K_1$ such that*

$$\alpha(x) = \sum_{\lambda\in K_1} m(\lambda x)\left[a_0^\lambda + \sum_{i=1}^{|K_0|-1} a_i^\lambda(x,\dots,x)\right], \quad x\in G,$$

$$\sum_{\mu\in K_0} a_k^\lambda(x,\dots,x,\underbrace{\mu y,\dots,\mu y}_{i}) = 0, \quad x,y\in G,\ \lambda\in K_1,\ 1\le i\le k<|K_0|,$$

$$\sum_{\lambda\in K_1} a_0^\lambda = 1,$$

(ii) *if $\mathbb{K} = \mathbb{R}$, then there exist $a_0^\lambda, b_0^\lambda \in \mathbb{R}$, k-additive, symmetric mappings $a_k^\lambda, b_k^\lambda\colon G^k\to\mathbb{R}$,$k\in\{1,\dots,|K_0|-1\}, \lambda\in K_1$ such that*

$$\alpha(x) = \sum_{\lambda\in K_1}\Bigg(Re\ (m(\lambda x))\left[a_0^\lambda + \sum_{i=1}^{|K_0|-1} a_i^\lambda(x,\dots,x)\right] \\ - Im\ (m(\lambda x))\left[b_0^\lambda + \sum_{i=1}^{|K_0|-1} b_i^\lambda(x,\dots,x)\right]\Bigg), \quad x\in G,$$

$$\sum_{\mu\in K_0} a_k^\lambda(x,\dots,x,\underbrace{\mu y,\dots,\mu y}_{i}) = 0, \quad x,y\in G,\ \lambda\in K_1,\ 1\le i\le k<|K_0|,$$

$$\sum_{\mu\in K_0} b_k^\lambda(x,\dots,x,\underbrace{\mu y,\dots,\mu y}_{i}) = 0, \quad x,y\in G,\ \lambda\in K_1,\ 1\le i\le k<|K_0|,$$

$$\sum_{\lambda\in K_1} a_0^\lambda = 1,$$

where $K_0 := \{\lambda\in K:\ m\circ\lambda = m\}, K_1$ is the set of representatives of cosets of the quotient group K/K_0.

Moreover, if α has the above form, then

$$\sum_{\lambda\in K}\alpha(\lambda x) = \sum_{\lambda\in K} m(\lambda x), \quad x\in G.$$

Proof. Assume that α satisfies (3). Let $\varphi\colon G\to\mathbb{K}$ be given by the formula

$$\varphi = \frac{1}{|K|}\sum_{\lambda\in K}\alpha\circ\lambda.$$

Then α and φ satisfy the equation

$$\frac{1}{|K|}\sum_{\lambda\in K}\alpha(x+\lambda y)=\varphi(y)\alpha(x),\quad x,y\in G.$$

In view of Theorem 2 we have the form of α such as in the statement of this corollary and the equality

$$\varphi(x)=\frac{1}{|K|}\sum_{\lambda\in K}m(\lambda x),\quad x\in G.$$

We observe that

(i) if $\mathbb{K}=\mathbb{C}$, then

$$\begin{aligned}|K|\varphi(x)&=\sum_{\mu\in K}\alpha(\mu x)=\sum_{\mu\in K}\sum_{\lambda\in K_1}m(\lambda\mu x)\left[a_0^\lambda+\sum_{i=1}^{|K_0|-1}a_i^\lambda(\mu x,\ldots,\mu x)\right]\\&=\sum_{\mu\in K_1}\sum_{\lambda\in K_1}m(\lambda\mu x)\left[|K_0|a_0^\lambda+\sum_{i=1}^{|K_0|-1}\sum_{\sigma\in K_0}a_i^\lambda(\sigma\mu x,\ldots,\sigma\mu x)\right]\\&=\sum_{\mu\in K_1}\sum_{\lambda\in K_1}m(\lambda\mu x)|K_0|a_0^\lambda=\sum_{\mu\in K}m(\mu x)\sum_{\lambda\in K_1}a_0^\lambda,\quad x,y\in G,\end{aligned}$$

(ii) if $\mathbb{K}=\mathbb{R}$, then

$$\begin{aligned}|K|\varphi(x)&=\sum_{\mu\in K}\alpha(\mu x)=\sum_{\mu\in K}\sum_{\lambda\in K_1}\mathrm{Re}\,(m(\lambda\mu x))\left[a_0^\lambda+\sum_{i=1}^{|K_0|-1}a_i^\lambda(\mu x,\ldots,\mu x)\right]\\&\quad-\sum_{\mu\in K}\sum_{\lambda\in K_1}\mathrm{Im}\,(m(\lambda\mu x))\left[b_0^\lambda+\sum_{i=1}^{|K_0|-1}b_i^\lambda(\mu x,\ldots,\mu x)\right]\\&=\sum_{\mu\in K_1}\sum_{\lambda\in K_1}\mathrm{Re}\,(m(\lambda\mu x))\left[|K_0|a_0^\lambda+\sum_{i=1}^{|K_0|-1}\sum_{\sigma\in K_0}a_i^\lambda(\sigma\mu x,\ldots,\sigma\mu x)\right]\\&\quad-\sum_{\mu\in K_1}\sum_{\lambda\in K_1}\mathrm{Im}\,(m(\lambda\mu x))\left[|K_0|b_0^\lambda+\sum_{i=1}^{|K_0|-1}\sum_{\sigma\in K_0}b_i^\lambda(\sigma\mu x,\ldots,\sigma\mu x)\right]\\&=\sum_{\mu\in K_1}\sum_{\lambda\in K_1}\mathrm{Re}\,(m(\lambda\mu x))|K_0|a_0^\lambda-\sum_{\mu\in K_1}\sum_{\lambda\in K_1}\mathrm{Im}\,(m(\lambda\mu x))|K_0|b_0^\lambda\\&=\sum_{\mu\in K}\mathrm{Re}\,(m(\mu x))\sum_{\lambda\in K_1}a_0^\lambda-\sum_{\mu\in K}\mathrm{Im}\,(m(\mu x))\sum_{\lambda\in K_1}b_0^\lambda\\&=\sum_{\mu\in K}m(\mu x)\sum_{\lambda\in K_1}a_0^\lambda,\quad x,y\in G.\end{aligned}$$

Hence $\sum_{\lambda\in K_1} a_0^\lambda = 1$ and on the other hand a function α, which has the form such as in the statement of this corollary, satisfies Eq. (3). $\square$

Theorem 3. *Let functions $f\colon G\to X,\alpha\colon G\to\mathbb{K}$ be such that $\sum_{\lambda\in K} f\circ\lambda\neq 0$, $\alpha\neq 0,\sum_{\lambda\in K}\alpha\circ\lambda\neq |K|$. They satisfy the equation*

$$\sum_{\lambda\in K} f(x+\lambda y) = |K|f(x)+\alpha(x)\sum_{\lambda\in K} f(\lambda y),\quad x,y\in G, \tag{4}$$

if and only if there exist a homomorphism $m\colon G\to\mathbb{C}^*, A_0\in X$, k-additive, symmetric mappings $A_k\colon G^k\to X,k\in\{1,\dots,L-1\}$ such that

$$f(x) = A_0+\sum_{i=1}^{|K|-1} A_i(x,\dots,x)-\alpha(x)A_0,\quad x\in G, \tag{5}$$

$$\sum_{\mu\in K} A_k(x,\dots,x,\underbrace{\mu y,\dots,\mu y}_{i}) = 0,\quad x,y\in G,\ 1\le i\le k<|K|, \tag{6}$$

and

(i) *if $\mathbb{K}=\mathbb{C}$, then there exist $a_0^\lambda\in\mathbb{C}$, k-additive, symmetric mappings $a_k^\lambda\colon G^k\to\mathbb{C},k\in\{1,\dots,|K_0|-1\},\lambda\in K_1$ such that*

$$\alpha(x) = \sum_{\lambda\in K_1} m(\lambda x)\left[a_0^\lambda+\sum_{i=1}^{|K_0|-1} a_i^\lambda(x,\dots,x)\right],\quad x\in G, \tag{7}$$

$$\sum_{\mu\in K_0} a_k^\lambda(x,\dots,x,\underbrace{\mu y,\dots,\mu y}_{i}) = 0,\quad x,y\in G,\ \lambda\in K_1,\ 1\le i\le k<|K_0|, \tag{8}$$

$$\sum_{\lambda\in K_1} a_0^\lambda = 1, \tag{9}$$

(ii) *if $\mathbb{K}=\mathbb{R}$, then there exist $a_0^\lambda,b_0^\lambda\in\mathbb{R}$, k-additive, symmetric mappings $a_k^\lambda,b_k^\lambda\colon G^k\to\mathbb{R},k\in\{1,\dots,|K_0|-1\},\lambda\in K_1$ such that*

$$\alpha(x) = \sum_{\lambda\in K_1}\Bigg(Re\ (m(\lambda x))\left[a_0^\lambda+\sum_{i=1}^{|K_0|-1} a_i^\lambda(x,\dots,x)\right] \tag{10}$$

$$-Im\ (m(\lambda x))\left[b_0^\lambda+\sum_{i=1}^{|K_0|-1} b_i^\lambda(x,\dots,x)\right]\Bigg),\quad x\in G, \tag{11}$$

$$\sum_{\mu\in K_0} a_k^\lambda(x,\dots,x,\underbrace{\mu y,\dots,\mu y}_{i}) = 0,\quad x,y\in G,\ \lambda\in K_1,\ 1\le i\le k<|K_0|, \tag{12}$$

$$\sum_{\mu\in K_0} b_k^\lambda(x,\ldots,x,\underbrace{\mu y,\ldots,\mu y}_{i})=0,\quad x,y\in G,\ \lambda\in K_1,\ 1\le i\le k<|K_0|, \tag{13}$$

$$\sum_{\lambda\in K_1} a_0^\lambda=1, \tag{14}$$

where $K_0:=\{\lambda\in K:\ m\circ\lambda=m\}, K_1$ *is the set of representatives of cosets of the quotient group* K/K_0.

Moreover

$$\sum_{\lambda\in K} f(\lambda x)=\left(|K|-\sum_{\lambda\in K}\alpha(\lambda x)\right)A_0,\quad x\in G. \tag{15}$$

Proof. Let f and α satisfy (4). We observe that

$$\begin{aligned}\sum_{\mu\in K}\sum_{\lambda\in K} f(x+\lambda y+\mu z)&=|K|\sum_{\lambda\in K} f(x+\lambda y)\\ &\quad+\sum_{\lambda\in K}\alpha(x+\lambda y)\sum_{\mu\in K} f(\mu z)=|K|^2f(x)+|K|\alpha(x)\sum_{\lambda\in K}f(\lambda y)\\ &\quad+\sum_{\lambda\in K}\alpha(x+\lambda y)\sum_{\mu\in K}f(\mu z),\quad x,y,z\in G,\end{aligned}$$

and

$$\begin{aligned}\sum_{\mu\in K}\sum_{\lambda\in K} f(x+\lambda y+\mu z)&=\sum_{\mu\in K}\sum_{\lambda\in K} f(x+\lambda(y+\mu z))\\ &=|K|^2f(x)+\alpha(x)\sum_{\mu\in K}\sum_{\lambda\in K}f(\lambda y+\mu z)\\ &=|K|^2f(x)+|K|\alpha(x)\sum_{\lambda\in K}f(\lambda y)+\alpha(x)\sum_{\lambda\in K}\alpha(\lambda y)\sum_{\mu\in K}f(\mu z),\quad x,y,z\in G.\end{aligned}$$

Hence we have

$$\left(\sum_{\lambda\in K}\alpha(x+\lambda y)-\alpha(x)\sum_{\lambda\in K}\alpha(\lambda y)\right)\sum_{\mu\in K}f(\mu z)=0,\quad x,y,z\in G,$$

and we obtain that α satisfies Eq. (3). In view of Corollary 1 we get the form of α.

Now we notice that

$$\begin{aligned}|K|\sum_{\lambda\in K}f(\lambda x)+\sum_{\lambda\in K}\alpha(\lambda x)\sum_{\mu\in K}f(\mu y)&=\sum_{\mu\in K}\sum_{\lambda\in K}f(\lambda x+\mu y)\\ &=\sum_{\mu\in K}\sum_{\lambda\in K}f(\mu y+\lambda x)=|K|\sum_{\mu\in K}f(\mu y)+\sum_{\mu\in K}\alpha(\mu y)\sum_{\lambda\in K}f(\lambda x),\quad x,y\in G.\end{aligned}$$

Hence

$$\left(|K| - \sum_{\lambda\in K} \alpha(\lambda x)\right) \sum_{\mu\in K} f(\mu y) = \left(|K| - \sum_{\mu\in K} \alpha(\mu y)\right) \sum_{\lambda\in K} f(\lambda x), \quad x, y \in G.$$

which gives us Eq. (15) for some $A_0 \in X$. Now, we can write Eq. (4) in the form

$$\sum_{\lambda\in K} f(x+\lambda y) = |K|f(x) + \alpha(x)\left(|K| - \sum_{\lambda\in K} \alpha(\lambda y)\right) A_0, \quad x, y \in G.$$

Let $q\colon G \to X$ be given by the formula

$$q(x) = f(x) + \alpha(x)A_0, \quad x \in G.$$

Then from equalities (3), (4), (15) we have

$$\begin{aligned}\sum_{\lambda\in K} q(x+\lambda y) &= \sum_{\lambda\in K} f(x+\lambda y) + \sum_{\lambda\in K} \alpha(x+\lambda y)A_0 \\ &= |K|f(x) + \alpha(x)\sum_{\lambda\in K} f(\lambda y) + \alpha(x)\sum_{\lambda\in K} \alpha(\lambda y)A_0 \\ &= |K|f(x) + |K|\alpha(x)A_0 = |K|q(x), \quad x, y \in G.\end{aligned}$$

In view of Theorem 1 there exist $c \in X$, k-additive, symmetric mappings $A_k\colon G^k \to X$,$k \in \{1, \dots, |K|-1\}$ such that

$$\begin{aligned}&q(x) = c + \sum_{i=1}^{|K|-1} A_i(x, \dots, x), \quad x \in G, \\ &\sum_{\mu\in K} A_k(x, \dots, x, \underbrace{\mu y, \dots, \mu y}_{i}) = 0, \quad x, y \in G,\ 1 \le i \le k < |K|.\end{aligned}$$

Since

$$c = q(0) = f(0) + \alpha(0)A_0 = A_0,$$

we have

$$f(x) = A_0 + \sum_{i=1}^{|K|-1} A_i(x, \dots, x) - \alpha(x)A_0, \quad x \in G.$$

Now we assume that f satisfies conditions (5)–(6) and α satisfies conditions (7)–(9) in the complex case or (10)–(14) in the real case. In view of Theorem 1 a function $f + \alpha A_0$ satisfies Eq. (1) and in view of Corollary 1 α satisfies Eq. (3). We have

$$\begin{aligned}\sum_{\lambda\in K} f(x+\lambda y) &= \sum_{\lambda\in K} (f(x+\lambda y) + \alpha(x+\lambda y)A_0) - \sum_{\lambda\in K} \alpha(x+\lambda y)A_0 \\ &= |K|f(x) + |K|\alpha(x)A_0 - \alpha(x)\sum_{\lambda\in K} \alpha(\lambda y)A_0, \quad x, y \in G.\end{aligned}$$

Hence we obtain

$$\sum_{\lambda\in K} f(\lambda y) = |K|f(0) + |K|\alpha(0)A_0 - \alpha(0)\sum_{\lambda\in K}\alpha(\lambda y)A_0$$
$$= |K|A_0 - \sum_{\lambda\in K}\alpha(\lambda y)A_0, \quad x, y\in G$$

and

$$\sum_{\lambda\in K} f(x+\lambda y) = |K|f(x) + |K|\alpha(x)A_0 - \alpha(x)\sum_{\lambda\in K}\alpha(\lambda y)A_0$$
$$= |K|f(x) + \alpha(x)\sum_{\lambda\in K} f(\lambda y), \quad x, y\in G,$$

which ends the proof. □

Remark 1. Let $f\colon G\to X, \alpha\colon G\to\mathbb{K}, \alpha = 0$ or $\sum_{\lambda\in K} f\circ\lambda = 0$. Then they satisfy Eq. (15) if and only if f satisfies Eq. (1). Hence, in view of Theorem 1, we know the form of f.

Remark 2. Let $f\colon G\to X, \alpha\colon G\to\mathbb{K}, \sum_{\lambda\in K}\alpha\circ\lambda = |K|$. If they satisfy Eq. (15) then α satisfies Eq. (1) and we know its form. At the present moment we don't know the form of f.

Now we can prove the main theorem of this paper which is a pexiderized version of Theorem 3.

Theorem 4. *Let functions $f, g, h\colon G\to X, \alpha\colon G\to\mathbb{K}$ be such that $\sum_{\lambda\in K} f\circ\lambda\neq const$, $\sum_{\lambda\in K}\alpha\circ\lambda\neq const, \alpha(0)\neq 0$. They satisfy the equation*

$$\sum_{\lambda\in K} f(x+\lambda y) = |K|g(x) + \alpha(x)h(y), \quad x, y\in G, \tag{16}$$

if and only if there exist a homomorphism $m\colon G\to\mathbb{C}^*, A, B, A_0\in X$, k-additive, symmetric mappings $A_k\colon G^k\to X, k\in\{1,\ldots,|K|-1\}$ such that

$$f(x) = A + A_0 + \sum_{i=1}^{|K|-1} A_i(x,\ldots,x) - \frac{\alpha(x)}{\alpha(0)}A_0, \quad x\in G, \tag{17}$$

$$g(x) = A + A_0 + \sum_{i=1}^{|K|-1} A_i(x,\ldots,x) - \frac{\alpha(x)}{\alpha(0)}[A + A_0 - B], \quad x\in G, \tag{18}$$

$$h(x) = \frac{1}{\alpha(0)}\left[\left(|K| - \sum_{\lambda\in K}\frac{\alpha(\lambda x)}{\alpha(0)}\right)A_0 + |K|(A-B)\right], \quad x\in G, \tag{19}$$

$$\sum_{\mu\in K} A_k(x,\ldots,x,\underbrace{\mu y,\ldots,\mu y}_{i}) = 0, \quad x, y\in G,\ 1\le i\le k<|K|, \tag{20}$$

and

(i) *if* $\mathbb{K} = \mathbb{C}$, *then there exist* $a_0^\lambda \in \mathbb{C}$, *k-additive, symmetric mappings* $a_k^\lambda \colon G^k \to \mathbb{C}$, $k \in \{1, \ldots, |K_0| - 1\}$,$\lambda \in K_1$ *such that*

$$\alpha(x) = \alpha(0) \sum_{\lambda \in K_1} m(\lambda x) \left[a_0^\lambda + \sum_{i=1}^{|K_0|-1} a_i^\lambda(x, \ldots, x) \right], \quad x \in G, \tag{21}$$

$$\sum_{\mu \in K_0} a_k^\lambda(x, \ldots, x, \underbrace{\mu y, \ldots, \mu y}_{i}) = 0, \quad x, y \in G,\ \lambda \in K_1,\ 1 \le i \le k < |K_0|, \tag{22}$$

$$\sum_{\lambda \in K_1} a_0^\lambda = 1, \tag{23}$$

(ii) *if* $\mathbb{K} = \mathbb{R}$, *then there exist* $a_0^\lambda, b_0^\lambda \in \mathbb{R}$, *k-additive, symmetric mappings* $a_k^\lambda, b_k^\lambda \colon G^k \to \mathbb{R}$,$k \in \{1, \ldots, |K_0| - 1\}$,$\lambda \in K_1$ *such that*

$$\alpha(x) = \alpha(0) \sum_{\lambda \in K_1} \left(Re\ (m(\lambda x)) \left[a_0^\lambda + \sum_{i=1}^{|K_0|-1} a_i^\lambda(x, \ldots, x) \right] \right. \tag{24}$$

$$\left. - Im\ (m(\lambda x)) \left[b_0^\lambda + \sum_{i=1}^{|K_0|-1} b_i^\lambda(x, \ldots, x) \right] \right), \quad x \in G, \tag{25}$$

$$\sum_{\mu \in K_0} a_k^\lambda(x, \ldots, x, \underbrace{\mu y, \ldots, \mu y}_{i}) = 0, \quad x, y \in G,\ \lambda \in K_1,\ 1 \le i \le k < |K_0|, \tag{26}$$

$$\sum_{\mu \in K_0} b_k^\lambda(x, \ldots, x, \underbrace{\mu y, \ldots, \mu y}_{i}) = 0, \quad x, y \in G,\ \lambda \in K_1,\ 1 \le i \le k < |K_0|, \tag{27}$$

$$\sum_{\lambda \in K_1} a_0^\lambda = 1, \tag{28}$$

where $K_0 := \{\lambda \in K : \ m \circ \lambda = m\}, K_1$ *is the set of representatives of cosets of the quotient group* K/K_0.

Proof. Putting $x = 0$ in (16) we have

$$\sum_{\lambda \in K} f(\lambda y) = |K| g(0) + \alpha(0) h(y), \quad y \in G.$$

Putting $y = 0$ in (16) we get

$$|K| f(x) = |K| g(x) + \alpha(x) h(0), \quad x \in G.$$

Hence we get

$$g(x) = f(x) - \frac{\alpha(x)}{|K|}h(0) = f(x) - \frac{\alpha(x)}{\alpha(0)}[f(0) - g(0)], \quad x \in G, \tag{29}$$

$$h(y) = \frac{1}{\alpha(0)}\left[\sum_{\lambda\in K} f(\lambda y) - |K|g(0)\right], \quad y \in G. \tag{30}$$

Let $f_0 = f - f(0), \alpha_0 = \frac{\alpha}{\alpha(0)}$. From the above equalities we obtain

$$\begin{aligned}\sum_{\lambda\in K} f_0(x+\lambda y) &= \sum_{\lambda\in K} f(x+\lambda y) - |K|f(0) = |K|g(x) + \alpha(x)h(y) - |K|f(0)\\ &= |K|f_0(x) - \alpha_0(x)|K|[f(0)-g(0)] + \alpha_0(x)\left[\sum_{\lambda\in K} f(\lambda y) - |K|g(0)\right]\\ &= |K|f_0(x) + \alpha_0(x)\sum_{\lambda\in K} f_0(\lambda y), \quad x, y \in G.\end{aligned}$$

In view of Theorem 3 there exist a homomorphism $m\colon G \to \mathbb{C}^*, A_0 \in X$, k-additive, symmetric mappings $A_k\colon G^k \to X, k \in \{1,\ldots,|K|-1\}$ such that

$$f_0(x) = A_0 + \sum_{i=1}^{|K|-1} A_i(x,\ldots,x) - \alpha_0(x)A_0, \quad x \in G,$$

$$\sum_{\mu\in K} A_k(x,\ldots,x,\underbrace{\mu y,\ldots,\mu y}_{i}) = 0, \quad x, y \in G,\ 1 \le i \le k < |K|,$$

and

(i) if $\mathbb{K} = \mathbb{C}$, then there exist $a_0^\lambda \in \mathbb{C}$, k-additive, symmetric mappings $a_k^\lambda\colon G^k \to \mathbb{C}$, $k \in \{1,\ldots,|K_0|-1\}, \lambda \in K_1$ such that

$$\alpha_0(x) = \sum_{\lambda\in K_1} m(\lambda x)\left[a_0^\lambda + \sum_{i=1}^{|K_0|-1} a_i^\lambda(x,\ldots,x)\right], \quad x \in G,$$

$$\sum_{\mu\in K_0} a_k^\lambda(x,\ldots,x,\underbrace{\mu y,\ldots,\mu y}_{i}) = 0, \quad x, y \in G,\ \lambda \in K_1,\ 1 \le i \le k < |K_0|,$$

$$\sum_{\lambda\in K_1} a_0^\lambda = 1,$$

(ii) if $\mathbb{K} = \mathbb{R}$, then there exist $a_0^\lambda, b_0^\lambda \in \mathbb{R}$, k-additive, symmetric mappings $a_k^\lambda, b_k^\lambda \colon G^k \to \mathbb{R}, k \in \{1, \dots, |K_0| - 1\}, \lambda \in K_1$ such that

$$\alpha_0(x) = \sum_{\lambda \in K_1} \left(\text{Re } (m(\lambda x)) \left[a_0^\lambda + \sum_{i=1}^{|K_0|-1} a_i^\lambda(x, \dots, x) \right] \right.$$
$$\left. - \text{Im } (m(\lambda x)) \left[b_0^\lambda + \sum_{i=1}^{|K_0|-1} b_i^\lambda(x, \dots, x) \right] \right), \quad x \in G,$$

$$\sum_{\mu \in K_0} a_k^\lambda(x, \dots, x, \underbrace{\mu y, \dots, \mu y}_{i}) = 0, \quad x, y \in G, \ \lambda \in K_1, \ 1 \le i \le k < |K_0|,$$

$$\sum_{\mu \in K_0} b_k^\lambda(x, \dots, x, \underbrace{\mu y, \dots, \mu y}_{i}) = 0, \quad x, y \in G, \ \lambda \in K_1, \ 1 \le i \le k < |K_0|,$$

$$\sum_{\lambda \in K_1} a_0^\lambda = 1.$$

Moreover

$$\sum_{\lambda \in K} f_0(\lambda x) = \left(|K| - \sum_{\lambda \in K} \alpha_0(\lambda x) \right) A_0, \quad x \in G.$$

Hence, putting $A := f(0), B := g(0)$ and using equalities (29), (30), we obtain the form of f, g, h and α.

Now we assume that f, g, h satisfy conditions (17)–(20) and α satisfies conditions (21)–(23) in the complex case and (24)–(28) in the real case. In view of Theorem 1 a function $f + \alpha_0 A_0$ satisfies Eq. (1) and in view of Corollary 1 α_0 satisfies Eq. (3). We have

$$\sum_{\lambda \in K} f(x + \lambda y) = \sum_{\lambda \in K} (f(x + \lambda y) + \alpha_0(x + \lambda y) A_0) - \sum_{\lambda \in K} \alpha_0(x + \lambda y) A_0$$
$$= |K| f(x) + |K| \alpha_0(x) A_0 - \alpha_0(x) \sum_{\lambda \in K} \alpha_0(\lambda y) A_0 = |K| f(x)$$
$$- |K| \alpha_0(x) [A - B] + \alpha_0(x) \left[|K|(A - B) + |K| A_0 - \sum_{\lambda \in K} \alpha_0(\lambda y) A_0 \right]$$
$$= |K| g(x) + \alpha(x) h(y), \quad x, y \in G,$$

which ends the proof. □

Remark 3. Let $f, g, h \colon G \to X, \alpha \colon G \to \mathbb{K}$ satisfy Eq. (16).

(i) If $\alpha(0) = 0$, then $\sum_{\lambda \in K} f \circ \lambda = \text{const}$.

(ii) If $\sum_{\lambda \in K} f \circ \lambda = \text{const}$, then $\sum_{\lambda \in K} \alpha \circ \lambda = \text{const}$ or $h = \text{const}$ (in this case Eq. (16) becomes Eq. (1) and we know its form). We don't know the form of the solutions in the case when $\sum_{\lambda \in K} \alpha \circ \lambda = \text{const}$.

References

[1] Acél, J., Chung, J.K., Ng, C.T.: Symmetric Second Differences in Product form on Groups. Topics in Mathematical Analysis, 1–22, Series in Pure Mathematics, vol. 11. World Scientific Publishing, Teaneck (1989)
[2] Badora, R.: On a generalized Wilson functional equation. Georgian Math. J. **12**(4), 595–606 (2005)
[3] Förg-Rob, W., Schwaiger, J.: A generalization of the cosine equation to n summands. Grazer Math. Ber. **316**, 219–226 (1992)
[4] Gajda, Z.: A remark on the talk of W. Förg-Rob. Grazer Math. Ber. **316**, 234–237 (1992)
[5] Łukasik, R.: Some generalization of Cauchy's and the quadratic functional equations. Aequ. Math. **83**, 75–86 (2012)
[6] Łukasik, R.: Some generalization of the quadratic and Wilson's functional equation. Aequ. Math. doi:10.1007/s00010-013-0185-y
[7] Stetkær, H.: On a signed cosine equation of N summands. Aequ. Math. **51**(3), 294–302 (1996)
[8] Stetkær, H.: Wilson's functional equation on $\mathbb{C}$. Aequ. Math. **53**(1-2), 91–107 (1997)
[9] Stetkær, H.: Functional equation on abelian groups with involution. Aequ. Math. **54**(1–2), 144–172 (1997)
[10] Stetkær, H.: Functional equations involving means of functions on the complex plane. Aequ. Math. **56**, 47–62 (1998)

Radosław Łukasik
Institute of Mathematics
University of Silesia
ul. Bankowa 14
40-007 Katowice
Poland
e-mail: rlukasik@math.us.edu.pl

Received: July 30, 2013
Revised: November 4, 2013